Stability criterion of coupled soliton states

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By operator theory, we prove that the stability of coupled fundamental soliton solutions of two coupled nonlinear Schrödinger equations is determined by $dP/d\beta$ criterion (with P the power or energy and β the propagation constant). Examples of the application of the stability criterion to the coupled fundamental soliton states in nonlinear couplers, birefringent fibers, and birefringent nonlinear planar waveguides are given. The predictions from the analytical stability criterion are consistent with numerical results. [S1063-651X(98)03503-X]

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I. INTRODUCTION

Optical solitons have been the subject of great interest over last several decades following the works of Refs. [1], [2], and [3]. Either in view of understanding their dynamic evolution or in view of their practical application, the stability of solitons (which are defined here as stationary objects) of a nonlinear system is a critical property. Indeed, much effort has been devoted to the study of soliton stability [4–20]. In general, the stability of soliton solutions of a nonlinear system requires a case by case analysis, very often resorting to a numerical approach [7–19]. Remarkably, however, it was shown [4–6] that there exists a stability criterion concerning the sign of $dP/d\beta$ for the fundamental bright soliton solutions of the scalar nonlinear Schrödinger equation that involves one wave, as was first demonstrated for the solitons in saturable nonlinear media [4], later extended to fundamental nonlinear modes in nonlinear waveguide structures [5], and further developed to the fundamental solitary waves in media with arbitrary nonlinearity [6]. In various situations, a nonlinear system may involve two or more waves (pulses) such as nonlinear couplers, birefringent fibers, and birefringent nonlinear planar waveguides. Naturally, it is interesting and important to find out whether $dP/d\beta$ criterion can be extended to the stability of the fundamental solitons of the coupled nonlinear Schrödinger equations governing two or more waves (or pulses). To address this issue, we conducted an analytical stability analysis. We prove by operator theory that the stability of the coupled fundamental soliton states of the coupled nonlinear Schrödinger equations involving two waves or pulses is determined by the sign of $dP/d\beta$. Examples of the application of the stability criterion to the coupled fundamental soliton states in nonlinear couplers [17], birefringent fibers [18,19], and birefringent nonlinear planar waveguides [20] show that the predictions from the analytical stability criterion are consistent with numerical calculations.

II. COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

We consider the coupled mode or coupled nonlinear Schrödinger equations governing the evolution of two waves or pulses in a general form,

$$i\,\frac{\partial u}{\partial z}+\frac{1}{2}\,\frac{\partial^2 u}{\partial T^2}+bu+\kappa v+n_2(|u|^2+\alpha|v|^2)u+n_2\,\eta v^2u^*=0, \tag{1a}$$

$$i\,\frac{\partial v}{\partial z}+\frac{1}{2}\,\frac{\partial^2 v}{\partial T^2}-bv+\kappa u+n_2(\alpha|u|^2+|v|^2)v+n_2\,\eta u^2v^*=0, \tag{1b}$$

where $\alpha(\geqslant 0)$ is the cross phase modulation coefficient, the terms involving $\eta(\geqslant 0)$ account for nonlinear coupling, $n_2>0$ refers to the self focusing nonlinearity, $\kappa(\geqslant 0)$ is the linear coupling coefficient governing the coupling of the two waves or pulses, and $2b$ is the normalized birefringence or the propagation constant difference of the two waves. For a birefringent nonlinear planar waveguide [20], u and v in Eqs. (1) refer to the amplitudes of x and y polarizations with $\alpha=A$, $\eta=A/2$, $n_2=1$, $b\sim(\beta_x-\beta_y)/2$, and $\kappa=0$. In the context of a birefringent fiber, u and v can represent either the pulsed wave amplitudes of the x and y linear polarizations with $\alpha=\frac{2}{3}$, $\eta=\frac{1}{3}$, $n_2=1$, $b\neq 0$, and $\kappa=0$, or the pulsed wave amplitudes of the left and right circular polarizations with $\alpha=2$, $\eta=0$, $n_2=\frac{2}{3}$, and $b=0$ and the nonzero coupling coefficient κ proportional to the birefringence $(\beta_x-\beta_y)/2$ [18,19,21]. For a nonlinear coupler, u and v may stand for either the pulsed wave amplitudes of the symmetric and antisymmetric modes with $\alpha=2$, $\eta=1$, $n_2=1/2$, $b(=C)\neq 0$, and $\kappa=0$, or pulsed wave amplitudes in individual waveguides 1 and 2 with $n_2=1$ and $b=0$, negligible cross phase modulation $\alpha\cong 0$, and negligible nonlinear coupling $\eta\cong 0$ but $\kappa(=C)\neq 0$ [22–24]. Because of their conservative nature, the general form of the coupled nonlinear Schrödinger equations (1) has three invariants, namely, the power (for spatial solitons) or energy (for temporal solitons)

$$P=\int_{-\infty}^{\infty}(|u|^2+|v|^2)dT, \tag{2a}$$

the momentum

$$M=\frac{i}{2}\int_{-\infty}^{\infty}\left(u\,\frac{du^*}{dT}-u^*\,\frac{du}{dT}+v\,\frac{dv^*}{dT}-v^*\,\frac{dv}{dT}\right)dT, \tag{2b}$$

and the Hamiltonian

$$H=\int_{-\infty}^{\infty}\left\{\frac{1}{2}\left(\left|\frac{du}{dT}\right|^2+\left|\frac{dv}{dT}\right|^2\right)-b(|u|^2-|v|^2)\right.$$
$$-n_2\left[\frac{|u|^4+|u|^4}{2}+\alpha|u|^2|v|^2+\kappa(uv^*+vu^*)\right.$$
$$\left.\left.+\frac{\eta}{2}(v^2u^{*2}+u^2v^{*2})\right]\right\}dT. \tag{2c}$$

III. STABILITY ANALYSIS

The coupled nonlinear Schrödinger equations (1) admit a set of bright stationary solutions of the form $u_n(T,z)=u_{ns}(T)\exp(i\beta z+i\phi_u)$ and $v_n(T,z)=v_{ns}(T)\exp(i\beta z+i\phi_v)$ $(n=1,2,3,...)$, with $u_{ns}(T)$ and $v_{ns}(T)$ being the solutions of the coupled ordinary differential equations

$$\frac{1}{2}\frac{\partial^2 u_{ns}}{\partial T^2}-(\beta-b)u_{ns}+d_{l,e}\kappa v_{ns}+n_2[u_{ns}{}^2+(\alpha\pm\eta)v_{ns}{}^2]u_{ns}=0, \tag{3a}$$

$$\frac{1}{2}\frac{\partial^2 v_{ns}}{\partial T^2}-(\beta+b)v_{ns}+q_{l,e}\kappa u_{ns}+n_2[(\alpha\pm\eta)u_{ns}{}^2+v_{ns}{}^2]v_{ns}=0, \tag{3b}$$

where $d_l=q_l=1$, along with the upper sign, corresponds to $\phi_u-\phi_v=0$ of the linear polarization, and $d_e=q_e=\mp i$ (for $\kappa=0$ when u_{ns} and v_{ns} are real) and the lower sign are for $\phi_u-\phi_v=\pm\pi/2$ of the elliptical polarization. These stationary solutions are usually obtainable only numerically except for some special cases. In the following, however, we will consider the stability of the fundamental (lowest order $n=1$) soliton states of the stationary solutions characterized by one hump in each amplitude profile of u_{ns} and v_{ns} with no phase (or π phase for $\kappa=0$) difference between the two components.

To reveal the stability character of the stationary soliton solution $\mathbf{U}=[u,v]_t=\mathbf{S}=[u_{1s},v_{1s}]_t$ of Eqs. (3), we can either examine the invariant $H=H+\beta P$ or analyze the linearized equations

$$i\frac{\partial\delta u}{\partial z}+\frac{1}{2}\frac{\partial^2\delta u}{\partial T^2}-(\beta-b)\delta u+d_{l,e}\kappa\delta v+(2|u_{1s}|^2+\alpha|v_{1s}|^2)\delta u+u_{1s}^2\delta u^*+\alpha u_{1s}(v_{1s}^*\delta v+v_{1s}\delta v^*)\pm\eta v_{1s}(2u_{1s}^*\delta v+v_{1s}\delta u^*)=0, \tag{4a}$$

$$i\frac{\partial\delta v}{\partial z}+\frac{1}{2}\frac{\partial^2\delta v}{\partial T^2}-(\beta+b)\delta v+q_{l,e}\kappa\delta u+(\alpha|u_{1s}|^2+2|v_{1s}|^2)\delta v+v_{1s}^2\delta v^*+\alpha v_{1s}(u_{1s}^*\delta u+u_{1s}\delta u^*)\pm\eta u_{1s}(2v_{1s}^*\delta u+u_{1s}\delta v^*)=0, \tag{4b}$$

obtained upon substitution of $u=(u_{1s}+\delta u)\exp(i\beta z+i\phi_u)$ and $v=(v_{1s}+\delta v)\exp(i\beta z+i\phi_v)$ into Eqs. (1). This set of linearized equations, in terms of the real and imaginary parts $\delta u=a_1+ib_1$ and $\delta v=a_2+ib_2$ $[a_{1,2}\sim\exp(\gamma z)$ and $b_{1,2}\sim\exp(\gamma z)]$, has forms of

$$-\gamma\mathbf{A}=\mathbf{L}_0\mathbf{B}, \tag{5a}$$

$$\gamma\mathbf{B}=\mathbf{L}_1\mathbf{A}, \tag{5b}$$

with $\mathbf{A}=[a_1,a_2]_t$, $\mathbf{B}=[b_1,b_2]_t$, subscript t referring to transpose, and self adjoint operators

$$\mathbf{L}_0=\begin{bmatrix}\frac{1}{2}\frac{\partial^2}{\partial T^2}-\beta+b+n_2[u_{1s}^2+(\alpha\mp\eta)v_{1s}^2], & d_{l,e}\kappa\pm 2n_2\eta u_{1s}v_{1s}\\ q_{l,e}\kappa\pm 2n_2\eta u_{1s}v_{1s}, & \frac{1}{2}\frac{\partial^2}{\partial T^2}-\beta-b+n_2[v_{1s}^2+(\alpha\mp\eta)u_{1s}^2]\end{bmatrix}, \tag{5c}$$

$$\mathbf{L}_1=\begin{bmatrix}\frac{1}{2}\frac{\partial^2}{\partial T^2}-\beta+b+n_2[3u_{1s}^2+(\alpha\pm\eta)v_{1s}^2], & d_{l,e}\kappa+2n_2(\alpha\pm\eta)u_{1s}v_{1s}\\ q_{l,e}\kappa+2n_2(\alpha\pm\eta)u_{1s}v_{1s}, & \frac{1}{2}\frac{\partial^2}{\partial T^2}-\beta-b+n_2[3v_{1s}^2+(\alpha\pm\eta)u_{1s}^2]\end{bmatrix}. \tag{5d}$$

If real positive $\gamma>0$ of Eqs. (5a) and (5b) for solutions $\mathbf{A}$ and $\mathbf{B}$ exists, the stationary state $\mathbf{S}$ is unstable. Otherwise, it is stable. This is equivalent to the invariant

$$H_N=H+\beta P, \tag{6}$$

being a maximum $(\delta^2H_N<0)$ or minimum $(\delta^2H_N>0)$ at $\mathbf{U}=[u,v]_t=\mathbf{S}=[u_{1s},v_{1s}]_t$, since the stationary solutions of Eqs. (3) are those at which $\delta H_N=0$. In terms of the real part of the perturbation functions $\mathbf{A}$, the linearized equations (5a) and (5b) read

$$\mathbf{L}_0\mathbf{L}_1\mathbf{A}=-\gamma^2\mathbf{A}. \tag{5e}$$

For the fundamental solitons $\mathbf{S}=[u_{1s},v_{1s}]_t$, operator $\mathbf{L}_1$ has at least one positive eigenvalue [because $\mathbf{L}_1(d\mathbf{S}/dT)=0$ derived by taking derivative of Eqs. (3) with respect to T, and $d\mathbf{S}/dT$ is not the fundamental eigenfunction of $\mathbf{L}_1$], whereas $\mathbf{L}_0$ has zero as its largest eigenvalue with the fundamental eigenfunction $\mathbf{S}$ [i.e., $L_0\mathbf{S}=0$ of Eqs. (3)] and all the other eigenvalues of $\mathbf{L}_0$ are negative. This fundamental eigenfunction $\mathbf{S}$ of $\mathbf{L}_0$ is orthogonal to $\mathbf{A}$ as $\gamma\langle\mathbf{S}_t|\mathbf{A}\rangle=-\langle\mathbf{S}_t|L_0\mathbf{B}\rangle$

$=-\langle \mathbf{B}_t | \mathbf{L}_0 \mathbf{S} \rangle = 0$, where $\langle \mathbf{G}_t | \mathbf{K} \rangle = \int_{-\infty}^{\infty} \mathbf{G}_t \mathbf{K}\, dT$. Thus, so far as the solutions $\mathbf{A}$ with $\gamma \neq 0$ are concerned, Eq. (5) needs to be solved only in the subspace orthogonal to $\mathbf{S}$. In this subspace the inverse operator $\mathbf{L}_0^{-1}$ exists; and for the fundamental soliton state $\mathbf{S}=[u_{1s}, v_{1s}]_t$, the operator $\mathbf{L}_0(\mathbf{L}_0^{-1})$ is negative definite (because all the eigenvalues of $\mathbf{L}_0$ or $\mathbf{L}_0^{-1}$ are nonpositive). The variational principle can then be applied for obtaining the largest value γ^2 of Eq. (5e)

$$\gamma^2 = \max \frac{\langle \mathbf{A}_t | \mathbf{L}_1 \mathbf{A} \rangle}{-\langle \mathbf{A}_t | \mathbf{L}_0^{-1} \mathbf{A} \rangle}. \tag{7}$$

Since the denominator $-\langle \mathbf{A}_t | L_0^{-1} \mathbf{A} \rangle$ is a positive quantity ensured by the negativeness of $\mathbf{L}_0^{-1}$, the value of the numerator of Eq. (7) $G = \max \langle \mathbf{A}_t | \mathbf{L}_1 \mathbf{A} \rangle$ decides whether there exists an exponential growth $\gamma^2 > 0$ from a perturbation for the fundamental soliton state **S.** If $G > 0$, a real $\gamma > 0$ exists, implying that the fundamental soliton state is unstable; $G < 0$ means that γ is imaginary, and no exponential growth results from a perturbation. With the help of the Lagrange multiplier technique, together with spectral analysis (with details delegated to Appendix A), we find that the sign $\gamma^2 \sim G = \max \langle \mathbf{A}_t | \mathbf{L}_1 \mathbf{A} \rangle$, and that the stability of the fundamental solitons of the coupled nonlinear Schrödinger equations is determined by the number of the positive eigenvalues of operator $\mathbf{L}_1$ and the sign of $dP/d\beta$. If $\mathbf{L}_1$ has one and only one positive eigenvalue, $dP/d\beta > 0$ $[P = \int_{-\infty}^{\infty} (|u_{1s}|^2 + |v_{1s}|^2) dT]$ corresponds to the stable soliton state $G \sim \gamma^2 < 0$, whereas $dP/d\beta < 0$ is associated with an unstable soliton state $(G \sim \gamma^2 > 0)$. On the other hand, if $\mathbf{L}_1$ has two (or more) positive eigenvalues, $G \sim \gamma^2 > 0$ and the soliton state $\mathbf{S}=[u_{1s}, v_{1s}]_t$ is unstable.

This stability criterion involving the number of the positive eigenvalues $\mathbf{L}_1$ along with the sign of $dP/d\beta$ for the stable soliton state $\mathbf{S}$ is, in fact, a direct consequence of the normalized Hamiltonian H_N of Eq. (6) being a local minimum at $\mathbf{U}=\mathbf{S}$. The reason is that the normalized Hamiltonian H_N at $\mathbf{S}$ for a stable state achieves a local minimum $\delta H_N = 0$. If

$$\delta^2 H_N = -\langle \mathbf{B}_t | L_0 \mathbf{B} \rangle - \langle \mathbf{A}_t | L_1 \mathbf{A} \rangle,$$

derived by substituting $u = (u_{1s} + a_1 + ib_1)\exp(i\beta z + i\phi_u)$ and $v = (v_{1s} + a_2 + ib_2)\exp(i\beta z + i\phi_v)$ into Eq. (6), is positive, $\mathbf{S}$ is stable. Otherwise, it is unstable. Since $\mathbf{L}_0$ for the fundamental solitons is negative definite, $-\langle \mathbf{B} | L_0 \mathbf{B} \rangle$ is positive. Also, $G = \max \langle \mathbf{A}_t | \mathbf{L}_1 \mathbf{A} \rangle < 0$ or $-\langle \mathbf{A} | \mathbf{L}_1 \mathbf{A} \rangle > 0$ when $\mathbf{L}_1$ has only one positive eigenvalue and $dP/d\beta > 0$, as aforementioned. We then conclude that $\delta^2 H_N > 0$, corresponding to a local minimum of H_N or a stable soliton state, when $dP_1/d\beta > 0$ and $\mathbf{L}_1$ has one positive eigenvalue.

This stability criterion derived above for the fundamental soliton states of the coupled nonlinear Schrödinger equations simplifies the stability problem from solving the coupled ordinary differential equations of Eqs. (5a) and (5b) involving operators $\mathbf{L}_0$ and $\mathbf{L}_1$ to the determination of the number p of the positive eigenvalues of operator $\mathbf{L}_1$ and the sign of $dP/d\beta$. The number p can usually be derived analytically either by solving the eigenvalue problem $\mathbf{L}_1 \mathbf{h} = \lambda \mathbf{h}$ directly or by examining at limiting points the positive eigenvalues of $\mathbf{L}_1$ along with the fact that the number of the positive eigenvalues of $\mathbf{L}_1$ changes only at a bifurcation (see Appendix B). These details will be illustrated in the following examples.

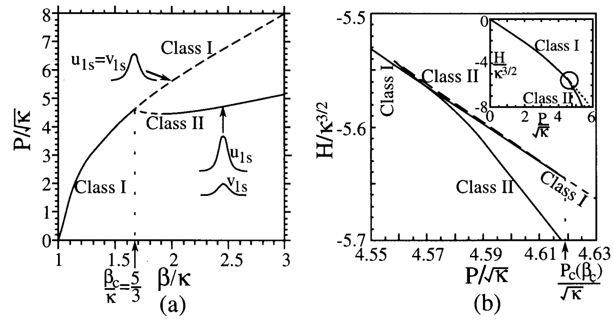

FIG. 1. Energy P vs propagation constant β in (a), and H of Eq. (2c) versus P in (b), for a nonlinear coupler.

IV. STABILITY OF FUNDAMENTAL SOLITON STATES IN NONLINEAR COUPLERS

We first consider the stability of the fundamental soliton states in nonlinear couplers as an example of application of the stability criterion. For a nonlinear coupler, the analysis can be based on either the composite symmetric and antisymmetric modes of $u = u_s$ and $v = u_a$, or modes of individual waveguides of $u = u_1$ and $v = u_2$ which are related to the former by $u_s = (u_1 + u_2)/\sqrt{2}$ and $u_a = (u_1 - u_2)/\sqrt{2}$ [25]. In terms of the symmetric and antisymmetric modes, the parameters in Eqs. (1) take $\alpha = 2$, $\eta = 1$, $n_2 = \frac{1}{2}$, $b = C$, and $\kappa = 0$, and with $u = u_1$, $v = u_2$, $\alpha = \eta = b = 0$, $n_2 = 1$, and $\kappa = C$ in Eqs. (1). By setting u, $v = u_s$, u_a, or u_1 u_2, Eqs. (3) can be shown to support two families of the fundamental soliton sates. One class of the stationary solutions has the analytical form

$$u_{1s} = 2\sqrt{\beta - \kappa}\, \mathrm{sech}[\sqrt{2(\beta - \kappa)}T], \quad v_{1s} = 0. \tag{8a}$$

in terms of the symmetric and antisymmetric modes u_s and u_a, or

$$u_{1s} = v_{1s} = \sqrt{2(\beta - \kappa)}\mathrm{sech}[\sqrt{2(\beta - \kappa)}T] \tag{8b}$$

in terms of u_1 and u_2, which can survive above $\beta > \kappa$. The other class of stationary solutions has nonzero or different magnitudes in both components, and can only be obtained numerically [see, e.g., the inset of Fig. 1(a)]. This class of solutions exists above a certain critical value $\beta = \beta_c$, and it bifurcates from the family of the soliton solutions of Eq. (8) at $\beta = \beta_c$, as shown in energy $P[= \int_{-\infty}^{\infty} (|u_{1s}|^2 + |v_{1s}|^2) dT]$ versus β diagram of Fig. 1(a), where the family of solutions (8) is referred to as class I and the other one is identified as class II in the figure. The bifurcation occurs at $\beta = \beta_c = \frac{5}{3}\kappa$, which can be derived from a perturbation method.

For class II solutions with nonzero or unequal magnitudes in both components that bifurcates from class I solutions at $\beta = \beta_c$, operator $\mathbf{L}_1$ has only one positive eigenvalue. $dP/d\beta > 0$ then indicates stable soliton sates, corresponding to the solid curve of Fig. 1 (class II), and $dP/d\beta < 0$ implies unstable soliton sates, identified by the dashed curve in the figure (class II). This conclusion on the stability of class II soliton solutions is consistent with the numerical stability analysis [17].

The fact that $\mathbf{L}_1$ for class II soliton solutions has one positive eigenvalue here is derived by recognizing that (a) the number of the positive eigenvalue of $\mathbf{L}_1$ can change only through a bifurcation (see Appendix B) and there is no bifurcation over its existence region between $\beta=\beta_c$ and $\beta\to\infty$ [Fig. 1(a)], and (b) $\mathbf{L}_1$ has only one positive eigenvalue at $\beta\to\infty$. At $\beta\to\infty$, $u_1\neq 0$ and $u_2\to 0$ or $u_1\to 0$ and $u_2\neq 0$, and operator $\mathbf{L}_1\sim\beta\left[\begin{smallmatrix}L_{10} & 0\\ 0 & L_{11}\end{smallmatrix}\right]$ or $\mathbf{L}_1\sim\beta\left[\begin{smallmatrix}L_{11} & 0\\ 0 & L_{10}\end{smallmatrix}\right]$, with $L_{10}=[\partial^2/\partial(\sqrt{2\beta}T)^2]-1+6\,\mathrm{sech}^2(\sqrt{2\beta}T)$ and $L_{11}=[\partial^2/\partial(\sqrt{2\beta}T)^2]-1$. Because of its diagonal, the eigenvalues of operator $\mathbf{L}_1$ are a superposition of the eigenvalues of L_{10} and L_{11}. By directly solving the eigenvalue problem $L_{10,11}f=\lambda_{10,11}f$ analytically, or by borrowing the result from the analysis for linear waveguides [26], L_{10} is found to have only one positive eigenvalue $\lambda_{10p}=3$ with the second eigenvalue $\lambda_{10s}=0$, and L_{11} has no positive eigenvalue. Operator $\mathbf{L}_1$ then has only one positive eigenvalue $\lambda_p=3\beta$ at $\beta\to\infty$, and consequently has one positive eigenvalue over its whole existence region.

For the class I soliton solutions of Eq. (8), $\mathbf{L}_1$ has only one positive eigenvalue as well below the bifurcation value $\beta<\beta_c$; and accordingly $dP/d\beta>0$ means that this class of soliton states is stable for $\beta<\beta_c$ (marked by the solid curve in Fig. 1). This conclusion of one positive eigenvalue of operator $\mathbf{L}_1$ below $\beta<\beta_c$ for class I soliton solutions is derived by directly examining the eigenvalue problem $\mathbf{L}_1\mathbf{f}=\lambda\mathbf{f}$ with $\mathbf{L}_1=\left[\begin{smallmatrix}L_{1s} & 0\\ 0 & L_{1a}\end{smallmatrix}\right]$, $L_{1s}=\frac{1}{2}(\partial^2/\partial T^2)-\beta+\kappa+6(\beta-\kappa)\mathrm{sech}^2[\sqrt{2(\beta-\kappa)}\,T]$ and $L_{1a}=\frac{1}{2}(\partial^2/\partial T^2)-\beta-\kappa+6(\beta-\kappa)\mathrm{sech}^2[\sqrt{2(\beta-\kappa)}\,T]$. For any $\beta(>\kappa)$, L_{1s} is found analytically to have one positive eigenvalue $\lambda_{1sp}=3(\beta-\kappa)$, and its second eigenvalue $\lambda_{1s0}=0$. The largest eigenvalue of L_{1a} is $\lambda_{1aL}=3\beta-5\kappa$, which is negative below $\beta<\beta_c=\frac{5}{3}\kappa$. Therefore, $\mathbf{L}_1$ has only one positive eigenvalue $\lambda_{1p}=3(\beta-\kappa)$ below bifurcation $\beta<\beta_c$.

On the other hand, above the bifurcation $\beta>\beta_c=\frac{5}{3}\kappa$, $\lambda_{1aL}=3\beta-5\kappa>0$; operator $\mathbf{L}_1$ for the class I soliton solutions has two positive eigenvalues $\lambda_{1p}=\lambda_{1sp}$ and λ_{1aL}. Consequently, class I soliton states become unstable above the bifurcation $\beta>\beta_c$, as identified by the dashed curve in Fig. 1. These conclusions from the stability criterion for class I soliton states again agree with the results from numerical calculations [17].

V. STABILITY OF FUNDAMENTAL SOLITON STATES IN BIREFRINGENT FIBERS

A second example of the application of the stability criterion is the stability of the fundamental soliton states in birefringent fibers. In terms of x and y linear polarizations $u=u_x$ and $v=u_y$ with $\alpha=\frac{2}{3}$, $\eta=\frac{1}{3}$, $n_2=1$, $b=\Delta\beta$, and $\kappa=0$, or circular polarizations $u=u_r=(u_x+iu_y)/\sqrt{2}$ and $v=u_l=(u_x-iu_y)/\sqrt{2}$ with $\alpha=2$, $\eta=0$, $n_2=2/3$, $b=0$, and $\kappa=\Delta\beta$, Eqs. (1) describe the pulse evolution in birefringent fibers. For this case, Eqs. (1) admit two families of the fundamental soliton states. One is the slow soliton

$$u_{1s}=\sqrt{2(\beta-\kappa)}\mathrm{sech}[\sqrt{2(\beta-k)}T],\quad v_{1s}=0 \tag{9a}$$

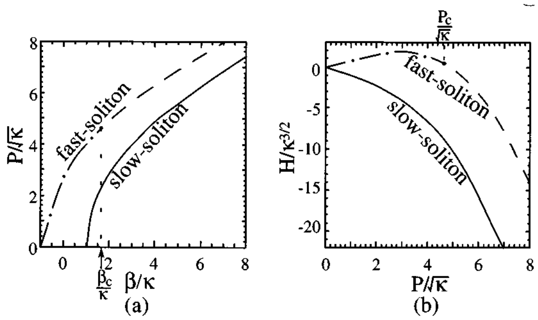

FIG. 2. Energy P vs propagation constant β in (a), and H of Eq. (2c) vs P in (b), for a birefringent fiber. The solid curves represent stable solitons, dashed curves unstable soliton states, and dashed-dotted curves higher order soliton states.

for $\beta>\kappa$, and the other is the fast soliton

$$u_{1s}=0,\quad v_{1s}=\sqrt{2(\beta+\kappa)}\mathrm{sech}[\sqrt{2(\beta+\kappa)}T] \tag{9b}$$

for $\beta>\beta_c=1.53\kappa$, in terms of the linear polarizations. In terms of the circular polarizations, the solutions are

$$u_{1s}=v_{1s}=\sqrt{\beta-\kappa}\mathrm{sech}[\sqrt{2(\beta-\kappa)}T] \tag{9a}$$

for the slow soliton, and

$$u_{1s}=-v_{1s}=\sqrt{\beta+\kappa}\ \mathrm{sech}[\sqrt{2(\beta+\kappa)}T] \tag{9b}$$

for the fast soliton. These are the fundamental states because $\mathbf{L}_0$ for the slow soliton above $\beta>\kappa$ and for the fast soliton above the bifurcation value $\beta>\beta_c=1.53\kappa$ (where higher order elliptically polarized solitons emerge) has no positive eigenvalue. The corresponding energy $P=2\sqrt{2(\beta-\kappa)}$ of the slow soliton and $P=2\sqrt{2(\beta+\kappa)}$ of the fast soliton are plotted graphically in Fig. 2.

For the fundamental solitons, we can apply the stability criterion. Following the analytical approach of Sec. IV, operator $\mathbf{L}_1$ for the slow soliton is found to have only one positive eigenvalue in the region of its existence $(\beta>\kappa)$, but for the fast soliton it has two positive eigenvalues above $\beta>\beta_c$. Thus $dP/d\beta=\sqrt{2/(\beta-\kappa)}>0$ means that the fundamental slow soliton in the birefringent fiber is stable. On the other hand, the fundamental fast soliton above $\beta>\beta_c$ is unstable in spite of its $dP/d\beta>0$. These conclusions are consistent with the numerical stability analysis presented in Ref. [18].

VI. STABILITY OF SOLITON STATES IN BIREFRINGENT NONLINEAR WAVEGUIDES

Now we examine the stability of the fundamental soliton states in birefringent nonlinear planar waveguides for which the parameters in Eqs. (1) take $\alpha=A$, $\eta=A/2$, $n_2=1$, $b<0$ and $\kappa=0$ [20]. The stationary soliton states of corresponding Eqs. (3) can have TE type,

$$u_{1s}=\sqrt{2(\beta+|b|)}\mathrm{sech}[\sqrt{2(\beta+|b|)}T],\quad v_{1s}=0, \tag{10a}$$

TM type

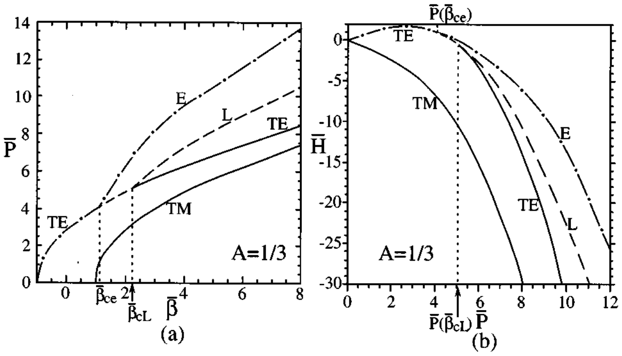

FIG. 3. Power $\overline{P}(=P/\sqrt{|b|})$ vs propagation constant $\overline{\beta}(=\beta/|b|)$ in (a), and $\overline{H}(=H/|b|^{3/2})$ of Eq. (2c) vs $\overline{P}$ in (b), for birefringent nonlinear planar waveguides with $A=\frac{1}{3}$. The solid curves represent stable solitons, dashed curves unstable soliton states, and dashed-dotted curves higher order soliton states.

$$u_{1s}=0,\quad v_{1s}=\sqrt{2(\beta-|b|)}\mathrm{sech}[\sqrt{2(\beta-|b|)}T],\tag{10b}$$

and linear and elliptical polarizations ($u_{1s}\neq 0$ and $v_{1s}\neq 0$) that are obtainable only numerically. The linear and elliptically polarized soliton states may bifurcate from either the TE or TM soliton, depending on the value A. For $0<A<2/3$, both linearly (L) and elliptically (E) polarized solitons branch out from the TE soliton as shown in Fig. 3 for $A=\frac{1}{3}$; when $\frac{2}{3}<A<2$, the elliptically polarized soliton state bifurcates from the TE soliton and the linearly polarized soliton state branches out from the TM soliton (see Fig. 4 for $A=1$); while $2<A<\infty$, both linearly and elliptically polarized solitons bifurcate from the TM soliton with the case of $A=4$ illustrated in Fig. 5. Although the exact soliton solutions with linear and elliptical polarizations require numerical approach, their bifurcation points can be derived analytically by a perturbation method [27]. The bifurcation of solitons with the linear polarization from the TM or TE soliton occurs at

$$\beta=\beta_{cL}=\pm\frac{(\sqrt{12A+1}-1)^2+4}{(\sqrt{12A+1}-1)^2-4}|b|,\tag{11}$$

and the elliptical polarized soliton state branches out from the TM or TE soliton at

$$\beta=\beta_{ce}=\pm\frac{(\sqrt{4A+1}-1)^2+4}{(\sqrt{4A+1}-1)^2-4}|b|.\tag{12}$$

These analytical expressions coincide with those from the numerical calculations.

To analyze the stability of the soliton states by the criterion derived in Sec. III, we first need to identify whether the soliton states in question are the fundamental soliton states, i.e., whether operator $\mathbf{L}_0$ of Eq. (5c) has zero as its largest eigenvalue, or whether

$$DD=-\langle \mathbf{A}_t|\mathbf{L}_0^{-1}\mathbf{A}\rangle\tag{13}$$

of the denominator of Eq. (7) is a positive quantity.

A. Stability of the TE soliton

For the TE soliton, $\mathbf{L}_0[\begin{smallmatrix} L_{0u} & 0\\ 0 & L_{0d}\end{smallmatrix}]$ is a diagonal operator. The eigenvalues of $L_{0u}=\frac{1}{2}(\partial^2/\partial T^2)-\beta-|b|+2(\beta+|b|)\mathrm{sech}^2\sqrt{2(\beta+|b|)}\,T$ and $L_{0d}=\frac{1}{2}(\partial^2/\partial T^2)-\beta+|b|+A(\beta+|b|)\mathrm{sech}^2\sqrt{2(\beta+|b|)}T$ determine the eigenvalues of $\mathbf{L}_0$. L_{0u} has the zero as its largest eigenvalue (since $L_{0u}u_{1s}=0$), and L_{0d} is found analytically to have its largest eigenvalue smaller than zero when $A<2$ and $\beta>\beta_{ce}$ (the bifurcation point for the elliptically polarized soliton). This means that the TE soliton is a fundamental soliton, and the stability criterion of Sec. III is applicable when $A<2$ and $\beta>\beta_{ce}$. For this TE soliton, operator $\mathbf{L}_1=[\begin{smallmatrix} L_{1u} & 0\\ 0 & L_{1d}\end{smallmatrix}]$ has two positive eigenvalues, for both $L_{1u}=\frac{1}{2}(\partial^2/\partial T^2)-\beta-|b|+6(\beta+|b|)\mathrm{sech}^2\sqrt{2(\beta+|b|)}T$ and $L_{1d}=\frac{1}{2}(\partial^2/\partial T^2)-\beta+|b|+3A(\beta+|b|)\mathrm{sech}^2\sqrt{2(\beta+|b|)}T$ have a positive eigenvalue, when $0<A<2/3$ for $\beta<\beta_{cL}$ (the bifurcation value of the linearly polarized soliton) and when $2/3<A<2$ for any β ($>-|b|$ above which the TE soliton exists). The TE soliton is then unstable within $0<A<2/3$ and $\beta_{ce}<\beta<\beta_{cL}$ (identified by the dashed curve in Fig. 3 for $A=\frac{1}{3}$), and within $\frac{2}{3}<A<2$ and $\beta>\beta_{ce}$ (see, e.g., Fig. 4 for $A=1$). On the other hand, within $0<A<\frac{2}{3}$ for $\beta>\beta_{cL}$ $\mathbf{L}_1$ has only one positive eigenvalue (as L_{1u} still has one positive eigenvalue, i.e., $\partial u_{1s}/\partial T=0$, but L_{0d} has no positive eigenvalue). $dP/d\beta$ $(=\sqrt{2/(\beta+|b|)})>0$ then indicates that the TE soliton is stable within $0<A<2/3$ and $\beta>\beta_{cL}$. This is shown graphically in Fig. 3 for $A=\frac{1}{3}$, marked by the solid curve. These predictions from stability criterion are confirmed by numerical simulations to Eqs. (1).

B. Stability of the TM soliton

Analytic analysis similar to that performed for the TE soliton reveals that operator $\mathbf{L}_0$ for the TM soliton has no positive eigenvalue, and it is thus the fundamental soliton within $0<A<2$ for any β ($>|b|$ above which the TM soliton exists), and within $2<A<\infty$ for $\beta<\beta_{ce}$. The stability of the TM soliton in these regions is then determined by the number of positive eigenvalues of $\mathbf{L}_1$ and the sign of $dP/d\beta$. Operator $\mathbf{L}_1$ for the TM soliton has only one positive eigenvalue in $0<A<\frac{2}{3}$ for any $\beta(>|b|)$ and in $\frac{2}{3}<A<\infty$ for $\beta<\beta_{cL}$, whereas it has two positive eigenvalues in $2/3<A<\infty$ for $\beta>\beta_{cL}$. We then conclude that $dP/d\beta$ $(=\sqrt{2/(\beta-|b|)})>0$ indicates the stable TM soliton when

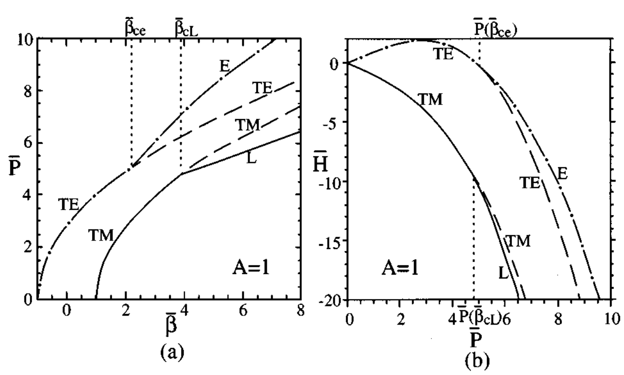

FIG. 4. Same as Fig. 3, but with $A=1$.

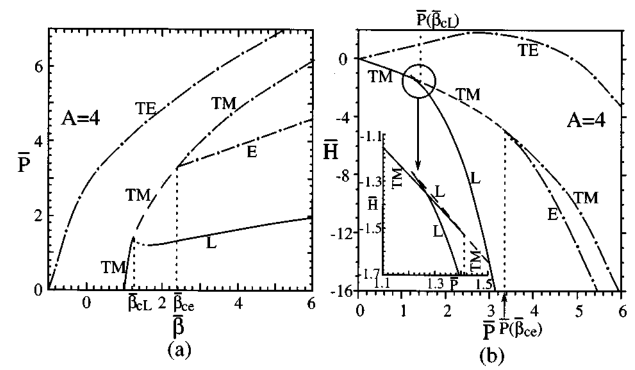

FIG. 5. Same as Fig. 3, but with $A=4$.

$0<A<2/3$ for any $\beta(>|b|)$ and when $2/3<A<\infty$ for $\beta<\beta_{cL}$, whereas the TM soliton is unstable when $2/3<A<2$ for any $\beta>\beta_{cL}$ and when $2<A<\infty$ for $\beta_{cL}<\beta<\beta_{ce}$. These distinctive stability characteristics of the TM soliton within three regions $0<A<2/3$, $2/3<A<2$ and $2<A<\infty$ are illustrated diagrammatically in Figs. 3, 4, and 5 for the cases of $A=\frac{1}{3}$, 1, and 4, where the dashed curves represent the unstable soliton and the solid curves stand for the stable soliton. Again, the predictions from the stability criterion conform to the numerical stability analysis of Eqs. (1).

C. Stability of linearly and elliptically polarized solitons

Operator $\mathbf{L}_0$ for the solitons with linear and elliptical polarizations is not diagonal; it is therefore convenient to identify the fundamental soliton states by examining the quantity DD of Eq. (13). Writing $\mathbf{Q}=L_0^{-1}\mathbf{A}$ and $\mathbf{A}=L_0\mathbf{Q}$, we have

$$DD=-\langle \mathbf{Q}_t|\mathbf{L}_0\mathbf{Q}\rangle. \tag{14}$$

Substitution of $\mathbf{Q}=[q_1,q_2]_t$ and $\mathbf{L}_0$ of Eq. (5c) into Eq. (14) produces

$$DD=\int_{-\infty}^{\infty}\left[\frac{1}{2}\left(u_{s1}\frac{\partial}{\partial T}\frac{q_1}{u_{s1}}\right)^2+\frac{1}{2}\left(v_{s1}\frac{\partial}{\partial T}\frac{q_2}{V_{s1}}\right)^2 \pm n_2A(v_{s1}q_1-u_{s1}q_2)^2\right]dT, \tag{15}$$

with the upper sign for the linear polarization and the lower sign for the elliptical polarization. Equation (15) indicates $DD>0$ ($n_2A>0$) for the linear polarization, whereas the positiveness of DD for the elliptical polarization is not guaranteed because of the minus sign in front of the last term. This means the linearly polarized soliton is a fundamental soliton, but the elliptically polarized one is not.

To ascertain the stability of the linearly polarized soliton by the stability criterion of Sec. III, we now must examine operator $\mathbf{L}_1$. By taking the limit of $\beta\to\infty$, we find analytically that $\mathbf{L}_1$ has two positive eigenvalues when $0<A<\frac{2}{3}$, and one positive eigenvalue when $2/3<A<\infty$. Since there is no bifurcation over $\beta_{cL}<\beta<\infty$, this means that over the whole region of its existence $\beta_{cL}<\beta<\infty$ $\mathbf{L}_1$ for the linearly polarized soliton has two positive eigenvalues when $0<A<2/3$ and one positive eigenvalue when $\frac{2}{3}<A<\infty$ (Appendix B). We then have the unstable linearly polarized soliton within $0<A<\frac{2}{3}$ for any $\beta>\beta_{cL}$, and with $A>\frac{2}{3}$ $dP/d\beta>0(<0)$ indicates a stable (unstable) linearly polarized soliton. The stability feature of the linearly polarized soliton within different regions $0<A<2/3$, $2/3<A<2$ and $2<A<\infty$ are shown in Figs. 3, 4, and 5 for $A=\frac{1}{3}$, 1, and 4. These predictions are consistent with numerical calculations.

VII. DISCUSSION

A. Conjectured stability criterion of minimum H of Eq. (2c)

In studying soliton states of birefringent nonlinear planar waveguides, Ref. [20] proposed a stability criterion by assuming that the soliton state with minimum of H of Eq. (2c) in the $H-P$ diagram is stable. This conjectured criterion happens to single out some stable soliton states, determined by the $dP/d\beta$ criterion along with the number of positive eigenvalues of $\mathbf{L}_1$ in the three examples given. However, for the fundamental solitons that do not have a minimum H in the $H-P$ diagram, the conjectured criterion gives either a false prediction or no prediction of the stability of the soliton states.

Consider the first example of the nonlinear coupler. The Hamiltonian H of Eq. (2c) versus energy P diagram of Fig. 1(b) for the nonlinear coupler indicates that the class I soliton has minimum H from $P/\sqrt{\kappa}=0$ to $P/\sqrt{\kappa}=4.569$ or from $\beta/\kappa=1$ to $\beta/\kappa=1.6524$ and above $P/\sqrt{\kappa}>4.569$ or $\beta/\kappa>1.9534$ the class II soliton state has a minimum H. This, according to the conjectured criterion, implies that the class I and II soliton states are stable in the corresponding parameter ranges, and that they can be unstable beyond the regions. However, based on the $dP/d\beta$ criterion of Sec. III in addition to these regions the class I soliton state is stable even above $\beta/\kappa>1.6524$ or $P/\sqrt{\kappa}>4.569$ up to the bifurcation point $\beta/\kappa=\beta_c/\kappa=5/3=1.6667$ or $P/\sqrt{\kappa}=4.619$, and the class II soliton state is stable even below $\beta/\kappa<1.9534$ or $P/\sqrt{\kappa}<4.569$, continuing to $\beta/\kappa=1.85$ or $P/\sqrt{\kappa}=4.555$, at which P is at a minimum. These stability characteristics of soliton states in the nonlinear coupler predicted by the $dP/d\beta$ criterion are completely in agreement with numerical stability analysis reported in Ref. [17].

Another example is birefringent nonlinear planar waveguides. Apart from elliptical polarization, soliton states in birefringent nonlinear planar waveguides for $A=4$ have the bifurcation characteristic of Fig. 5, similar to that in the nonlinear coupler of Fig. 1 (where the antisymmetric soliton corresponding to the TE soliton of Fig. 5 is not included). Between $P=0$ and 1.268 or $\beta=1$ and 1.201, the TM soliton has the minimum H of Eq. (2c), and above $P>1.268$ or $\beta>2.1$ the linearly polarized soliton has the minimum H. According to the conjectured criterion of minimum H, the TM soliton and the linearly polarized soliton are stable within the corresponding regions, and they can be unstable outside the regions. The $dP/d\beta$ criterion indicates that, in addition to these regions, the TM soliton is stable even beyond $\beta>1.201$ or $P>1.268$ up to the bifurcation $\beta=\beta_{cL}=1.25$ or $P=\sqrt{2}$, and the linearly polarized soliton state is stable even below $\beta<2.1$ or $P<1.268$ up to $\beta>1.6$ or $P=1.21$, at which P is at a minimum. These predictions accord with numerical stability analysis. The conjectured criterion of minimum H clearly cannot identify these stable branches that do not have a minimum H.

Similarly, according to the conjectured criterion, for $A=\frac{1}{3}$ only the TM soliton is stable [Fig. 3(b)], and for $A=1$ the TM soliton is stable below $\beta<\beta_{cL}$ and the linearly polarized soliton is stable above $\beta>\beta_{cL}$ [Fig. 4(b)]. These conclusions again happen to agree with those drawn from the stability criterion derived in Sec. III. Nevertheless, beyond these parameter ranges the criterion of minimum of the Hamiltonian H cannot conjecture the stability of the fundamental solitons, e.g., the TE soliton, which do not have minimum H, and may reckon them to be unstable. The present $dP/d\beta$ criterion can ascertain unequivocally that within $0<A<2/3$ the TE soliton is stable for $\beta>\beta_{cL}$ and is unstable for $\beta_{cL}>\beta>\beta_{ce}$ (see Fig. 3, with $A=\frac{1}{3}$), and that within $2/3<A<2$ the TE soliton is unstable for $\beta>\beta_{ce}$. In short, the criterion of minimum Hamiltonian H can possibly identify part of the stable branches predicted by the criterion derived in Sec. III, but it cannot reveal, or gives an incorrect prediction of, the stability characteristic of the fundamental solitons that do not have minimum H.

B. Stability near threshold and linearly growing perturbation

Based on the linear stability analysis, the sign of $dP/d\beta$ determines the stability of the fundamental solitons of the coupled nonlinear Schrödinger equations, when operator $\mathbf{L}_1$ has one positive eigenvalue. Although $dP/d\beta>0$ (<0) indicates a stable (unstable) soliton state, the degree of the stability (instability) may differ from point to point in the P-β curve. Near the threshold $dP/d\beta=0$, the soliton state associated with $dPd\beta>0$ may have a weak stability, and its stationary propagation can be destroyed by a large perturbation, as a consequence of shifting a soliton state from a stable region to an unstable region, because of its small stability region. Such shifting can possibly be caused by the existence, for any set of parameters, of linearly growing perturbation solutions

$$\mathbf{A}(T,z)=\mathbf{A}_1(T)z+\mathbf{A}_0(T),\quad \mathbf{B}(T,z)=\mathbf{B}_1(T)z+\mathbf{B}_0(T) \tag{16}$$

to the linearized equations

$$\frac{\partial \mathbf{A}}{\partial z}=-\mathbf{L}_0\mathbf{B},\quad \frac{\partial \mathbf{B}}{\partial z}=\mathbf{L}_1\mathbf{A}, \tag{17}$$

which may manifest in the absence of exponential growing perturbation (when $dP/d\beta>0$, and $\mathbf{L}_1$ has one positive eigenvalue). A perturbation of this type will not destroy the stationary propagation of the soliton, as in the case of the exponentially growing perturbation, but merely transform the soliton to a neighboring soliton state, similar to a single nonlinear Schrödinger equation [4]. In our analysis, we have ignored this type of trivial instability, as it is beyond the scope of the present paper.

VIII. CONCLUSIONS

We proved by operator theory that the stability of the coupled fundamental soliton states of the two coupled nonlinear Schrödinger equations is determined by the number of positive eigenvalues of operator $\mathbf{L}_1$ (which changes only at bifurcation) of the linearized equations and the sign of $dP/d\beta$. Examples of the application of the stability criterion to fundamental soliton states in nonlinear couplers, birefringent fibers, and birefringent nonlinear planar waveguides were given. The predictions from the stability criterion agreed with the numerical calculations.

APPENDIX A

We prove here that the sign of the numerator

$$\gamma^2\sim G=\max\langle \mathbf{A}_t|\mathbf{L}_1\mathbf{A}\rangle \tag{A1}$$

of Eq. (7) is decided by the number of positive eigenvalues of $\mathbf{L}_1$ and the sign of $dP/d\beta$. According to the method of indeterminate Lagrange multipliers, maximization of quantity $\langle \mathbf{A}_t|\mathbf{L}_1\mathbf{A}\rangle$ in Eq. (A1) is equivalent to solving the equation

$$\mathbf{L}_1\mathbf{F}=\lambda\mathbf{F}+q\mathbf{S} \tag{A2}$$

for the largest eigenvalue λ, which together with constant q is determined by the conditions of orthogonality $\langle \mathbf{F}_t|\mathbf{S}\rangle=0$ and normalization $\langle \mathbf{F}_t|\mathbf{F}\rangle=1$. Expanding

$$\mathbf{F}=\sum_{m=1}^{\infty} a_m\mathbf{F}_m, \tag{A3a}$$

$$\mathbf{S}=\sum_{m=1}^{\infty} c_m\mathbf{F}_m \tag{A3b}$$

of Eq. (A2) in the complete set of eigenfunctions $\mathbf{F}_m$ of the operator $\mathbf{L}_1$ gives rise to $\mathbf{F}=q\Sigma_{m=1}^{\infty}c_m\mathbf{F}_m/(\lambda_m-\lambda)$, with λ_m the eigenvalues of $\mathbf{L}_1$. This expansion $\mathbf{F}$, substituted into the orthogonality condition $\langle \mathbf{F}_t|\mathbf{S}\rangle=0$, produces an equation for determining λ,

$$qg(\lambda)=q\sum_{m=1}^{\infty}\frac{c_m^2}{\lambda_m-\lambda}=0, \tag{A4}$$

where $q=0$ only when $\lambda=\lambda_m$ with the eigenfunction of $\mathbf{L}_1$ orthogonal to $\mathbf{S}$. Equation (A4) indicates that the largest λ must either lie between the largest eigenvalue λ_{s1} and the second largest eigenvalue λ_{s2} of operator $\mathbf{L}_1$ with the eigenfunctions nonorthogonal to $\mathbf{S}$, or be equal to the largest eigenvalue λ_0 of $\mathbf{L}_1$ with the eigenfunction orthogonal to $\mathbf{S}$. That is, the number and character of the positive eigenvalues of $\mathbf{L}_1$ determine the sign of the largest eigenvalue λ. From Eqs. (3), we have $\mathbf{L}_1(d\mathbf{S}/dT)=0$. This means that $\mathbf{L}_1$ has a zero eigenvalue with the eigenfunction $d\mathbf{S}/dT$ that is not the fundamental eigenfunction of $\mathbf{L}_1$. Thus $\mathbf{L}_1$ has at least one positive eigenvalue for its fundamental eigenfunction nonorthogonal to $\mathbf{S}$. If $\mathbf{L}_1$ has one and only one positive eigenvalue, the sign of the largest $\lambda=\lambda_{\max}$, and consequently G and the stability is then determined by the sign of $g(0)$. This is because Eq. (A4) indicates that $\lambda_{\max}>0$ when $g(0)<0$, and $\lambda_{\max}<0$ when $g(0)>0$. $g(0)$ here is related to the power or energy $P=\int_{-\infty}^{\infty}(|u_{1s}|^2+|v_{1s}|^2)dT$ by

$$g(0)=\sum_{m=1}^{\infty}\frac{c_m^2}{\lambda_m}=\langle \mathbf{S}_t|L_1^{-1}\mathbf{S}\rangle=\langle \mathbf{S}_t|\partial\mathbf{S}/\partial\beta\rangle=\frac{1}{2}\frac{dP}{d\beta}, \tag{A5}$$

where expansion (A3b), along with its derivation

$$\sum_{m=1}^{\infty} \frac{c_m}{\lambda_m} \mathbf{F}_m = L_1^{-1}\mathbf{S}, \tag{A6}$$

was used in establishing the second equality, and the relation $\mathbf{L}_1 \partial \mathbf{S}/\partial\beta = \mathbf{S}$, obtained from differentiating equations (3), was invoked in establishing the third identity. In Eqs. (A5) and (A6), the inverse operator $\mathbf{L}_1^{-1}$ taken is justified by the fact that the expansion $\mathbf{S}$ in terms of eigenfunctions $\mathbf{F}_m$ does not involve eigenfunction $d\mathbf{S}/dT$ of $\mathbf{L}_1$ with zero eigenvalue $(\lambda_2=0)$ because of orthogonality $c_2=\langle \mathbf{S}_t | d\mathbf{S}/dT\rangle=0$. From Eq. (A5), it follows that $dP/d\beta>0$ corresponds to the stable soliton state as $g(0)>0$ gives $\lambda_{\max}<0$ $(G\sim\gamma^2<0)$, whereas $dP/d\beta<0$ is associated with an unstable soliton state because $g(0)<0$ yields $\lambda_{\max}>0$ $(G\sim\gamma^2>0)$.

On the other hand, if $\mathbf{L}_1$ has two (or more) positive eigenvalues, the largest λ must either lie between the two largest positive eigenvalue λ_{s1} and λ_{s2} of operator $\mathbf{L}_1$ with the eigenfunctions nonorthogonal to $\mathbf{S}$, or be equal to the largest positive eigenvalue λ_0 of $\mathbf{L}_1$ with the eigenfunction orthogonal to $\mathbf{S}$. In either case, the largest $\lambda=\lambda_{\max}$ must be positive; therefore $G\sim\gamma^2>0$ and the soliton state $\mathbf{S}=[u_{1s},v_{1s}]_t$ is unstable.

APPENDIX B

We show here that the number of positive eigenvalues $\mathbf{L}_1$ of the coupled nonlinear nonlinear Schrödinger equations changes at bifurcation where new families of soliton states emerge. When the number of positive eigenvalues of $\mathbf{L}_1$ changes, one of the eigenvalues of $\mathbf{L}_1$ must pass through zero, i.e., $\mathbf{L}_1$ has a zero eigenvalue $\mathbf{L}_1\mathbf{G}=0$. Proof of the change of the number of positive eigenvalues of $\mathbf{L}_1$ at bifurcation is then equivalent to prove that new families of soliton states emerge when a stationary soliton state $\mathbf{S}_0$, at which $\mathbf{L}_1\mathbf{G}=0$, changes to $\mathbf{S}_0+\delta\mathbf{S}$, as β varies from β_0 to $\beta_0+\delta\beta$.

We write soliton states $\mathbf{S}$ at β, slightly deviated from the state of $\mathbf{S}_0$ at β_0, in the form of

$$\mathbf{S}=\mathbf{S}_0+\delta\mathbf{S}, \tag{B1a}$$

$$\beta=\beta_0+\delta\beta, \tag{B1a}$$

which, substituted into Eqs. (3), give

$$\mathbf{L}_1\delta\mathbf{S}-\delta\beta\mathbf{S}_0=\delta\beta\,\delta\mathbf{S}-\mathbf{D}_1(\delta\mathbf{S})^2-\mathbf{D}_2(\delta\mathbf{S})^3-\mathbf{D}_3\delta\mathbf{S}_t\mathbf{J}\delta\mathbf{S} -\mathbf{D}_4\delta\mathbf{S}\delta\mathbf{S}_t\mathbf{J}\delta\mathbf{S}, \tag{B2}$$

where $\delta\mathbf{S}=\left[\begin{smallmatrix}\delta u_{s1}\\ \delta v_{s1}\end{smallmatrix}\right]$, $(\delta\mathbf{S})^2=\left[\begin{smallmatrix}\delta u_{s1}^2\\ \delta v_{s1}^2\end{smallmatrix}\right]$, $(\delta\mathbf{S})^3=\left[\begin{smallmatrix}\delta u_{s1}^3\\ \delta v_{s1}^3\end{smallmatrix}\right]$, $\mathbf{D}_1=n_2\left[\begin{smallmatrix}3u_{s1} & (\alpha\pm\eta)u_{s1}\\ (\alpha\pm\eta)v_{s1} & 3v_{s1}\end{smallmatrix}\right]$, $\mathbf{D}_2=n_2\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, $\mathbf{D}_3=n_2(\alpha\pm\eta)\mathbf{J}\mathbf{S}_0$, $\mathbf{D}_4=0.5n_2(\alpha\pm\eta)\mathbf{J}$, $\mathbf{S}_0=\left[\begin{smallmatrix}u_{s1}\\ v_{s1}\end{smallmatrix}\right]$, and $\mathbf{J}=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$. The terms on the right hand side of Eq. (B2) are of higher order, compared with those on the left hand side. If $\mathbf{L}_1$ does not have an eigenvalue with zero eigenvalue at $\mathbf{S}_0(\beta_0)$, $\delta\mathbf{S}$ is qualitatively the same as $\mathbf{S}$. From Eq. (B2), this solution $\delta\mathbf{S}$, to the first order, is of the form

$$\delta\mathbf{S}=\delta\beta\mathbf{L}_1^{-1}\mathbf{S}_0. \tag{B3}$$

On the other hand, if $\mathbf{L}_1$ has a zero eigenvalue $\mathbf{L}_1\mathbf{G}=0$ at $\mathbf{S}_0(\beta_0)$, the solution to Eq. (B2), to first order, is

$$\delta\mathbf{S}=\delta\beta\mathbf{L}_1^{-1}\mathbf{S}_0+\sigma\mathbf{G}, \tag{B4}$$

with σ a constant. Multiplying Eq. (B2) by $\mathbf{G}$, and performing subsequent integration, we have

$$Q=-\delta\beta\int \mathbf{G}_t\mathbf{S}_0\,dT=\int \mathbf{G}_t\Phi\; dT, \tag{B5}$$

where Φ represents all the higher order terms on the right hand side of Eq. (B2). Since terms $-\delta\beta\mathbf{S}_0$ and Φ are of different orders, the equality of Eq. (B5) is meaningful only when the quantity $Q=0$. This means $\mathbf{G}$ is orthogonal to $\mathbf{S}_0$ and orthogonal to $\Psi=\delta\beta\mathbf{L}_1^{-1}\mathbf{S}_0$. On the other hand, substitution of $\delta\mathbf{S}=\Psi+\sigma\mathbf{G}=\delta\beta\mathbf{L}_1^{-1}\mathbf{S}_0+\sigma\mathbf{G}$ into $Q=\int\mathbf{G}_t\Phi\,dT=0$ yields

$$\int_{-\infty}^{\infty}\mathbf{G}_t[\delta\beta(\Psi+\sigma\mathbf{G})-\mathbf{D}_1(\Psi+\sigma\mathbf{G})^2-\mathbf{D}_2(\Psi+\sigma\mathbf{G})^3 -\mathbf{D}_3(\Psi+\sigma\mathbf{G})_t\mathbf{J}(\Psi+\sigma\mathbf{G}) -\mathbf{D}_4(\Psi+\sigma\mathbf{G})(\Psi+\sigma\mathbf{G})_t\mathbf{J}(\Psi+\sigma\mathbf{G})]dT=0\,. \tag{B6}$$

Equation (B6) is a cubic polynomial equation yielding up to three solutions of σ, and consequently up to three solutions $\delta\mathbf{S}$. We then have bifurcation. This indicates that bifurcation occurs when $\mathbf{L}_1$ has a zero eigenvalue, i.e., when $\mathbf{L}_1$ changes the number of positive (negative) eigenvalues.

Consider examples in the text. $\mathbf{S}_0$ is a symmetric (even) function, and $\mathbf{G}$ is an antisymmetric (odd) function. Equation (B6) gives $\sigma=0$ and $\delta\mathbf{S}=\delta\beta\mathbf{L}_0^{-1}\mathbf{S}$ for one soliton state, and

$$\sigma^2\cong\frac{\delta\beta\int_{-\infty}^{\infty}\mathbf{G}_t[\mathbf{G}-2\mathbf{D}_1(\mathbf{G}\mathbf{L}_1^{-1}\mathbf{S}_0)-2\mathbf{D}_3(\mathbf{G}_t\mathbf{J}\mathbf{L}_1^{-1}\mathbf{S}_0)]dT}{\int_{-\infty}^{\infty}\mathbf{G}_t[\mathbf{D}_2\mathbf{G}^3+\mathbf{G}\mathbf{D}_4(\mathbf{G}_t\,\mathbf{J}\mathbf{G})]dT}$$

and $\delta\mathbf{S}=\delta\beta\mathbf{L}_1^{-1}\mathbf{S}+\sigma\mathbf{G}$ for the other bifurcating soliton solutions.

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